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Baxter's method for XXZ model

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Abstract. It is shown that Baxter's method when applied to the XXZ hamiltonian gives a set of eigenvectors and wavenumbers different from those of the familiar Bethe solution. In this limiting case of Baxter's results for the eight-vertex model we find that one of Baxter's parameters corresponds to the freedom of making arbitrary rotations about the z axis. It is suggested that for the general XYZ hamiltonian there may be a generalization of the ordinary z component of spin which would be conserved in addition to Baxter's quantum number.

1. Introduction

Recently Baxter has obtained the eigenvectors and eigenvalues both of the transfer matrix for the eight-vertex model (Baxter 1973a, b, c) and of the XYZ hamiltonian for a one-dimensional anisotropic Heisenberg chain of spins $\frac{1}{2}$ (Baxter 1972). The XYZ hamiltonian may be written

$$\mathscr{H}_{XYZ} = -\tfrac{1}{2}\sum_{j=1}^{N} [\tfrac{1}{2}(1+\Gamma)\sigma_j^x\sigma_{j+1}^x + \tfrac{1}{2}(1-\Gamma)\sigma_j^y\sigma_{j+1}^y + \tfrac{1}{2}\Delta\sigma_j^z\sigma_{j+1}^z], \tag{1.1}$$

where the σ_j are Pauli matrices with periodic boundary conditions, $\sigma_{j+N} = \sigma_j$, and the number N of sites in the chain is assumed to be even. The method of diagonalization consisted of finding invariant subspaces of $\mathscr{H}_{XYZ}$ labelled by an integer valued quantum number n and then constructing eigenvectors in each subspace. This procedure is precisely analogous to Bethe's method (Bethe 1931, Yang and Yang 1966) for the XXZ hamiltonian,

$$\mathscr{H}_{XXZ} = -\tfrac{1}{2}\sum_{j=1}^{N} \mathscr{H}_{j,j+1}, \tag{1.2a}$$

$$\mathscr{H}_{j,j+1} = \tfrac{1}{2}(\sigma_j^x\sigma_{j+1}^x + \sigma_j^y\sigma_{j+1}^y + \Delta\sigma_j^z\sigma_{j+1}^z), \tag{1.2b}$$

where the total z component of spin

$$S_z = \tfrac{1}{2}\sum_{j=1}^{N} \sigma_j^z \tag{1.3}$$

is conserved and hence is used to label invariant subspaces in which the diagonalization may be performed. Baxter has remarked however (Baxter 1973a) that in the $\Gamma \to 0$ limit of his solution one does not directly obtain the familiar Bethe solution for $\mathscr{H}_{XXZ}$. Our aim here is to study this limiting case in more detail. We have already examined Baxter's solution in the case $\Delta = 0$ (Jones 1973) and shown that it is not identical with the usual solution for the XY model. We will show here that similarly in the XXZ model Baxter's method may be applied to give a diagonalization of $\mathscr{H}_{XXZ}$ different from the ordinary Bethe solution. A study of this limiting case throws some light on the significance of the free parameters that occur in Baxter's general eigenstates. Also we find that the wavenumbers associated with Baxter's 'spin' waves are not the usual wavenumbers characteristic of a simple system of N spins.

Thus in § 2 we construct families of vectors which form invariant subspaces with respect to $\mathscr{H}_{XXZ}$. In § 3 we use a Bethe-type *ansatz* to construct eigenstates of $\mathscr{H}_{XXZ}$ within each allowed subspace. Finally we discuss the role of the free parameter that occurs in these new eigenstates.

2. Baxter families for $\mathscr{H}_{XXZ}$

We wish now to show briefly how to construct special families of vectors, invariant under the action of $\mathscr{H}_{XXZ}$, which are the $\Gamma \to 0$ limiting cases of Baxter's families (Baxter 1973b) for the eight-vertex model transfer matrix. Rather than directly take the limit of Baxter's results as stated for the transfer matrix, we will indicate how to derive the vectors using an argument directly fitted to the XXZ hamiltonian. This argument closely parallels that given for the XY model (Jones 1973).

Thus on each spin site j we define orthonormal pairs of spinors labelled by an integer l_j. We write the 'up' spinor as

$$\phi_{l_j,l_j+1} = (1+|p(l_j)|^2)^{-1/2}\begin{pmatrix} p(l_j) \\ 1 \end{pmatrix}, \tag{2.1a}$$

and the 'down' spinor as

$$\phi_{l_j,l_j-1} = (1+|p(l_j)|^2)^{-1/2}\begin{pmatrix} 1 \\ -p^*(l_j) \end{pmatrix}, \tag{2.1b}$$

where $p(l_j)$ is to be determined. Consider two adjacent spin sites $j, j+1$ with 'up' spinors on each site and the integers related by $l_{j+1} = l_j+1$. Since the spin space for the two sites is four dimensional we may write

$$\begin{aligned}\mathscr{H}_{j,j+1}\phi_{l_j,l_j+1} \otimes \phi_{l_j+1,l_j+2} &= D_1\phi_{l_j,l_j+1} \otimes \phi_{l_j+1,l_j+2} + D_2\phi_{l_j,l_j-1} \otimes \phi_{l_j+1,l_j+2} \\ &\quad + D_3\phi_{l_j,l_j+1} \otimes \phi_{l_j+1,l_j} + D_4\phi_{l_j,l_j-1} \otimes \phi_{l_j+1,l_j},\end{aligned} \tag{2.2}$$

where

$$D_4 = [(1+|p(l_j)|^2)(1+|p(l_j+1)|^2)]^{-1/2}[-p^2(l_j) - p^2(l_j+1) + 2\Delta p(l_j)p(l_j+1)]. \tag{2.3}$$

When working with the hamiltonian rather than with the transfer matrix, the quantities $p(l_j)$ are determined by the requirement that D_4 should vanish. The vanishing of D_4 has the consequence that the pair hamiltonian $\mathscr{H}_{j,j+1}$ does not simultaneously turn 'down' ('up') two adjacent 'up' ('down') spinors. Such a condition is essential in order that the direct product states which we define below (2.10a) form families which are closed under the action of the total hamiltonian $\mathscr{H}$. It does not seem possible to achieve this closure property by any other choice for D_4. Requiring that D_4 should vanish gives then the condition

$$p^2(l_j)+p^2(l_j+1) = 2\Delta p(l_j)p(l_j+1). \tag{2.4}$$

This equation enables us to find $p(l+1)$ once $p(l)$ is given. Henceforth let us work in the regime

$$-1 \leqslant \Delta \leqslant +1, \tag{2.5a}$$

which means we may parametrize Δ by

$$\Delta = \cos 2\eta, \tag{2.5b}$$

with real η. We may now solve (2.4) to get

$$p(l_j+1) = \mathrm{e}^{\pm i2\eta}p(l_j). \tag{2.6}$$

This result suggests that a convenient choice for $p(l)$ is

$$p(l) = p(l,s) = \mathrm{e}^{\mathrm{i}(s+2l\eta)}, \tag{2.7}$$

where s is assumed to be real and represents a freedom of phase in fixing $p(l)$. This s is the same as Baxter's parameter s in the eight-vertex model solution (Baxter 1973a, b, c). With this choice for $p(l,s)$ and defining

$$B(l,s) = \tfrac{1}{2}\mathrm{i}\sin 2\eta\, \mathrm{e}^{\mathrm{i}(s+2l\eta)}, \tag{2.8}$$

one may show that (2.2) becomes

$$\begin{aligned}\mathscr{H}_{j,j+1}\phi_{l_j,l_j+1} \otimes \phi_{l_j+1,l_j+2} &= \tfrac{1}{2}\Delta\phi_{l_j,l_j+1} \otimes \phi_{l_j+1,l_j+2} + B(l_j,s)\phi_{l_j,l_j-1} \otimes \phi_{l_j+1,l_j+2} \\ &\quad - B(l_j+1,s)\phi_{l_j,l_j+1} \otimes \phi_{l_j+1,l_j}.\end{aligned} \tag{2.9a}$$

Similarly (Jones 1973) one may show that

$$\begin{aligned}\mathscr{H}_{j,j+1}\phi_{l_j,l_j+1} \otimes \phi_{l_j+1,l_j} &= -\tfrac{1}{2}\Delta\phi_{l_j,l_j+1} \otimes \phi_{l_j+1,l_j} + B(l_j,s)\phi_{l_j,l_j-1} \otimes \phi_{l_j+1,l_j} \\ &\quad - B^*(l_j+1,s)\phi_{l_j,l_j+1} \otimes \phi_{l_j+1,l_j+2} + \phi_{l_j,l_j-1} \otimes \phi_{l_j-1,l_j},\end{aligned} \tag{2.9b}$$

$$\begin{aligned}\mathscr{H}_{j,j+1}\phi_{l_j,l_j-1} \otimes \phi_{l_j-1,l_j} &= -\tfrac{1}{2}\Delta\phi_{l_j,l_j-1} \otimes \phi_{l_j-1,l_j} + B^*(l_j,s)\phi_{l_j,l_j+1} \otimes \phi_{l_j-1,l_j} \\ &\quad - B(l_j-1,s)\phi_{l_j,l_j-1} \otimes \phi_{l_j-1,l_j-2} + \phi_{l_j,l_j+1} \otimes \phi_{l_j+1,l_j},\end{aligned} \tag{2.9c}$$

$$\begin{aligned}\mathscr{H}_{j,j+1}\phi_{l_j,l_j-1} \otimes \phi_{l_j-1,l_j-2} &= \tfrac{1}{2}\Delta\phi_{l_j,l_j-1} \otimes \phi_{l_j-1,l_j-2} + B^*(l_j,s)\phi_{l_j,l_j+1} \otimes \phi_{l_j-1,l_j-2} \\ &\quad - B^*(l_j-1,s)\phi_{l_j,l_j-1} \otimes \phi_{l_j-1,l_j}.\end{aligned} \tag{2.9d}$$

We may now define product states in the 2^N dimensional space W on which $\mathscr{H}_{XXZ}$ acts by (Baxter 1973a, b, c)

$$\psi(l_1,l_2,\ldots,l_N,l_{N+1}) = \phi_{l_1,l_2} \otimes \phi_{l_2,l_3} \otimes \ldots \otimes \phi_{l_{N-1},l_N} \otimes \phi_{l_N,l_{N+1}} \tag{2.10a}$$

where the sequence of integers $l_1,\ldots,l_{N+1}$ must obey certain constraints in order that families of such vectors should be closed under the action of $\mathscr{H}_{XXZ}$. The first constraint is

$$l_{j+1} = l_j \pm 1. \tag{2.10b}$$

The second constraint links together l_1 and l_{N+1} and is a kind of cyclic boundary condition. However, in order to impose this second constraint, it is necessary to restrict the values of η which occur in (2.5*b*) to be rational multiples of π. Precisely, we assume that there is an even integer L and an integer m_1 such that (Baxter 1973a)

$$L\eta = m_1\pi. \tag{2.10c}$$

With this definition of L we may write the second constraint as

$$l_{N+1} \equiv l_1 (\mathrm{mod}\ L). \tag{2.10d}$$

It is often convenient (Baxter 1973c) to denote $\psi(l_1, \ldots, l_{N+1})$ by giving the value of l_1 together with the sites $x_1 < x_2 < \ldots < x_n$ along the chain at which 'down' spinors occur in the product (2.10*a*). Writing $l_1 = l$,

$$\psi(l_1, l_2, \ldots, l_{N+1}) = \psi(l; x_1, x_2, \ldots, x_n), \tag{2.11a}$$

where in order to satisfy (2.10*b*, *d*) the number of 'down' spins n must satisfy

$$n \equiv \tfrac{1}{2}N (\mathrm{mod}\ \tfrac{1}{2}L). \tag{2.11b}$$

Because of (2.10*c*) we have that

$$\psi(l+L; x_1, \ldots, x_n) = \psi(l; x_1, \ldots, x_n). \tag{2.11c}$$

If one now combines together (1.2*a*, *b*) and (2.9*a*, *b*, *c*, *d*), one sees that the effect of $\mathscr{H}_{XXZ}$ on such a vector is to preserve the number n of 'down' spinors. However, the integer l may be unchanged or else shifted by ± 2 and an individual x_i may be unchanged or else shifted by ± 1. The set of vectors $\psi(l; x_1, \ldots, x_n)$ for n fixed but with the various choices of $x_1, \ldots, x_n$ and with l ranging from 0 or 1 in steps of two up to $L-1$ or L contains $\frac{1}{2}L\binom{N}{n}$ vectors. This set of vectors forms a family which is closed under the action of $\mathscr{H}_{XXZ}$. Such a family spans an invariant subspace labelled by the number n. Because l jumps in steps of 2, it is convenient to introduce an integer l_0 which may be either 0 or 1 and to write

$$\psi_j(x_1, \ldots, x_n) = \psi(l_0 + 2j - 2; x_1, \ldots, x_n), \tag{2.12a}$$

with (2.11*c*) becoming

$$\psi_{j+\frac{1}{2}L}(x_1, x_2, \ldots, x_n) = \psi_j(x_1, \ldots, x_n). \tag{2.12b}$$

When working out the action of $\mathscr{H}_{XXZ}$ on such states the coefficients in (2.9*a*, *b*, *c*, *d*) involving $B(l, s)$ all cancel out to give quite simple results. Thus if $N \equiv 0 (\mathrm{mod}\ L)$ there is a family of $n = 0$ states. For these we have

$$\mathscr{H}_{XXZ}\psi_j = -\tfrac{1}{4}N\Delta\psi_j. \tag{2.13}$$

If $N \equiv 2 (\mathrm{mod}\ L)$ there is an $n = 1$ family. For these one finds

$$\mathscr{H}_{XXZ}\psi_j(x) = -\tfrac{1}{4}N\Delta\psi_j(x) + \Delta\psi_j(x) - \tfrac{1}{2}\psi_j(x-1) - \tfrac{1}{2}\psi_j(x+1), \tag{2.14a}$$

where to include all cases we make the supplementary definitions

$$\psi_j(N+1) = \psi_{j+1}(1), \tag{2.14b}$$

$$\psi_j(0) = \psi_{j-1}(N). \tag{2.14c}$$

As a final example, if $N \equiv 4 (\mathrm{mod}\ L)$ there are $n = 2$ families allowed. If $x_2 \neq x_1+1$ one finds

$$\begin{aligned} \mathscr{H}_{XXZ}\psi_j(x_1, x_2) \\ &= -\tfrac{1}{4}N\Delta\psi_j(x_1, x_2) + 2\Delta\psi_j(x_1, x_2) - \tfrac{1}{2}\psi_j(x_1-1, x_2) \\ &\quad -\tfrac{1}{2}\psi_j(x_1+1, x_2) - \tfrac{1}{2}\psi_j(x_1, x_2-1) - \tfrac{1}{2}\psi_j(x_1, x_2+1), \end{aligned} \tag{2.15a}$$

while if $x_2 = x_1+1$

$$\begin{aligned} \mathscr{H}_{XXZ}\psi_j(x_1, x_1+1) \\ &= -\tfrac{1}{4}N\Delta\psi_j(x_1, x_1+1) + \Delta\psi_j(x_1, x_1+1) \\ &\quad -\tfrac{1}{2}\psi_j(x_1-1, x_1+1) - \tfrac{1}{2}\psi_j(x_1, x_1+2), \end{aligned} \tag{2.15b}$$

where we again make the supplementary definitions

$$\psi_j(x_1, N+1) = \psi_{j+1}(1, x_1) \tag{2.15c}$$

$$\psi_j(0, x_2) = \psi_{j-1}(x_2, N) \tag{2.15d}$$

3. Some eigenvectors of $\mathscr{H}_{XXZ}$

If we fix N and L, we may then expect to construct, for each value n allowed by (2.11b), eigenvectors of $\mathscr{H}_{XXZ}$ by taking a linear combination of the vectors in the family labelled by n. The working involved is quite like that for the ordinary Bethe solution (Bethe 1931) so we shall merely sketch the construction for the case $n = 2$ and then state the general result for other values of n.

Thus we look for a state

$$\Psi(2, s) = \exp[-\mathrm{i}\tfrac{1}{2}s(N-4)] \sum_{j=1}^{L/2} \sum_{x_1<x_2} f(j; x_1, x_2)\psi_j(x_1, x_2) \tag{3.1}$$

satisfying

$$\mathscr{H}_{XXZ}\Psi(2, s) = (\epsilon_2 - \tfrac{1}{4}N\Delta)\Psi(2, s). \tag{3.2}$$

The phase factor $\exp[-\mathrm{i}\frac{1}{2}s(N-4)]$ is chosen for later convenience in discussing the role of the parameter s and plays no part in the present calculation. We put the expression (3.1) into (3.2) and try to determine $f(j; x_1, x_2)$ and ϵ_2 by equating coefficients of $\psi_j(x_1, x_2)$ on each side of the resulting equation.

First look at terms $\psi_j(x_1, x_2)$ in which $x_1 \neq 1, x_2 \neq N$, and $x_2 \neq x_1+1$. By equating coefficients we obtain

$$\begin{aligned} \epsilon_2 f(j; x_1, x_2) &= 2\Delta f(j; x_1, x_2) - \tfrac{1}{2}f(j; x_1+1, x_2) - \tfrac{1}{2}f(j; x_1-1, x_2) \\ &\quad -\tfrac{1}{2}f(j; x_1, x_2+1) - \tfrac{1}{2}f(j; x_1, x_2-1). \end{aligned}$$

The form of this equation together with the general form of Baxter's results for the eight-vertex transfer matrix (Baxter 1973c) suggests trying a solution to this equation of the form

$$f(j;x_1,x_2)=C(q_1,q_2)h(q_1,j)h(q_2,j-1)\,e^{iq_1x_1}\,e^{iq_2x_2}$$
$$-C(q_2,q_1)h(q_2,j)h(q_1,j-1)\,e^{iq_2x_1}\,e^{iq_1x_2},$$

or using the notation $C_{12}=C(q_1,q_2)$,

$$h_1(j)=h(q_1,j),\ h_2(j)=h(q_2,j),$$

$$f(j;x_1,x_2)=C_{12}h_1(j)h_2(j-1)\,e^{iq_1x_1}e^{iq_2x_2}-C_{21}h_2(j)h_1(j-1)\,e^{iq_2x_1}\,e^{iq_1x_2}. \tag{3.3a}$$

The value of ϵ_2 is then

$$\epsilon_2=(\Delta-\cos q_1)+(\Delta-\cos q_2). \tag{3.3b}$$

Next equate coefficients of $\psi_j(1,x_2)$ when $2\leqslant j\leqslant \frac{1}{2}L$, $x_2\neq x_1+1$, $x_2\neq N$ to obtain

$$\epsilon_2 f(j;1,x_2)=2\Delta f(j;1,x_2)-\tfrac{1}{2}f(j;2,x_2)-\tfrac{1}{2}f(j-1;x_2,N)$$
$$-\tfrac{1}{2}f(j;1,x_2+1)-\tfrac{1}{2}f(j;1,x_2-1).$$

This equation is satisfied only if

$$f(j-1;x_2,N)=f(j;0,x_2),$$

which is equivalent to

$$C_{12}h_1(j)=-C_{21}h_1(j-2)\,e^{iq_1N} \tag{3.4}$$

together with the corresponding equation when q_1 and q_2 are interchanged.

If we look at coefficients of $\psi_j(x_1,x_1+1)$ for $x_1\neq 1, N-1$, we find

$$\epsilon_2 f(j;x_1,x_1+1)=\Delta f(j;x_1,x_1+1)-\tfrac{1}{2}f(j;x_1,x_1+2)-\tfrac{1}{2}f(j;x_1-1,x_1+1),$$

which is true only if

$$f(j;x_1,x_1)+f(j;x_1+1,x_1+1)=2\Delta f(j;x_1,x_1+1).$$

Substitution of the form (3.3*a*) leads to

$$\frac{C_{12}}{C_{21}}=\frac{h_1(j-1)}{h_1(j)}\frac{h_2(j)}{h_2(j-1)}\,e^{-i\Theta(q_1,q_2)}, \tag{3.5a}$$

where $\Theta(q_1,q_2)$ is the well known phase (Yang and Yang 1966) defined by

$$e^{-i\Theta(q_1,q_2)}=\frac{1+e^{i(q_1+q_2)}-2\Delta\,e^{iq_1}}{1+e^{i(q_1+q_2)}-2\Delta\,e^{iq_2}}. \tag{3.5b}$$

Finally from the terms $\psi_1(1,x_2)$ for $x_2\neq x_1+1, N$ we have

$$\epsilon_2 f(1;1,x_2)=2\Delta f(1;1,x_2)-\tfrac{1}{2}f(1;2,x_2)-\tfrac{1}{2}f(\tfrac{1}{2}L;x_2,N)$$
$$-\tfrac{1}{2}f(1;1,x_2+1)-\tfrac{1}{2}f(1;1,x_2-1)$$

which is possible only if

$$f(\tfrac{1}{2}L;x_2,N)=f(1;0,x_2).$$

This condition leads to

$$C_{12}h_1(1)h_2(0)=-C_{21}h_2(\tfrac{1}{2}L)h_1(\tfrac{1}{2}L-1)\,e^{iq_1N}, \tag{3.6}$$

and a similar equation with q_1, q_2 interchanged.

The aim next is to use equations (3.4), (3.5) and (3.6) to try to determine $C_{12}, h_1, h_2, q_1, q_2$. To do this let us make the reasonable hypothesis (Baxter 1973c) that the functions $h(j)$ have the same periodicity as $\psi_j(x_1, \cdots, x_n)$, namely we assume

$$h(j+\tfrac{1}{2}L) = h(j). \tag{3.7}$$

Now rewrite (3.5*a*) as

$$\frac{h_2(j)}{h_1(j)} = \frac{h_2(j-1)}{h_1(j-1)} \frac{C_{12}}{C_{21}} \mathrm{e}^{\mathrm{i}\Theta(q_1, q_2)}.$$

If we iterate this equation $L/2$ times and use the assumed periodicity we see that

$$\left(\frac{C_{12}}{C_{21}} \mathrm{e}^{\mathrm{i}\Theta(q_1, q_2)}\right)^{L/2} = 1.$$

We may satisfy this condition in the simplest way by choosing

$$\frac{C_{12}}{C_{21}} = \mathrm{e}^{-\mathrm{i}\Theta(q_1, q_2)}, \tag{3.8a}$$

$$C_{12} = \mathrm{e}^{-\frac{1}{2}\mathrm{i}\Theta(q_1, q_2)}. \tag{3.8b}$$

Having fixed C_{12}/C_{21} we now learn from (3.5*a*) that

$$\frac{h(q_2, j)}{h(q_2, j-1)} = \frac{h(q_1, j)}{h(q_1, j-1)} = \omega$$

where ω must be independent of either q_1 or q_2. It then follows that we may assume $h(q, j)$ to be independent of q and dependent only on j. If we look at (3.4) in the case that $L/2$ is an odd integer and use the periodicity of $h(j)$ we see that ω must also be independent of j. Thus a solution for $h(j)$ is

$$h(j) = \omega^{j-1}, \tag{3.9a}$$

where to satisfy the periodicity requirement (3.7) ω must be one of the $L/2$th roots of unity,

$$\omega^{L/2} = 1. \tag{3.9b}$$

The function $f(j; x_1, x_2)$ now may be written

$$f(j; x_1, x_2) = \sum_P \epsilon_P \exp\{-\tfrac{1}{2}\mathrm{i}\Theta(q_{P1}, q_{P2})\} g_{P1}(j, x_1) g_{P2}(j-1, x_2), \tag{3.10a}$$

with

$$g_r(j, x) = \omega^{j-1} \mathrm{e}^{\mathrm{i}q_r x}, \tag{3.10b}$$

where the sum is over permutations $P1, P2$ of the integers 1, 2 and ϵ_P is the parity of the permutation. Finally we see that (3.6) requires q_1, q_2 to satisfy the equations

$$\begin{aligned} \mathrm{e}^{\mathrm{i}q_1 N} \mathrm{e}^{\mathrm{i}\Theta(q_1, q_2)} &= -\omega^2, \\ \mathrm{e}^{\mathrm{i}q_2 N} \mathrm{e}^{\mathrm{i}\Theta(q_2, q_1)} &= -\omega^2. \end{aligned} \tag{3.10c}$$

One may now verify that all remaining conditions arising from coefficients of states $\psi_j(x_1, x_2)$ not yet examined are satisfied by the solution (3.10*a*, *b*) with wavenumbers (3.10*c*).

It is now straightforward to verify that from vectors $\psi_j(x_1, x_2, \ldots, x_n)$ in the family with n 'down' spins we can construct eigenvectors

$$\Psi(n; s) = \Psi(q_1, q_2, \ldots, q_n; s)$$
$$= \exp[-\mathrm{i}\tfrac{1}{2}s(N-2n)] \sum_{j=1}^{L/2} \sum_{x_1<x_2<\ldots<x_n} f(j; x_1, x_2, \ldots, x_n)\psi_j(x_1, \ldots, x_n), \tag{3.11a}$$

where

$$f(j, x_1, \ldots, x_n)$$
$$= \sum_P \epsilon_P \exp\left(-\tfrac{1}{2}\mathrm{i} \sum_{k<m} \Theta(q_{Pk}, q_{Pm})\right) g_{P1}(j, x_1) g_{P2}(j-1, x_2), \ldots, g_{Pn}(j-n+1, x_n), \tag{3.11b}$$

and the summation is over permutations $P1, P2, \ldots, Pn$ of the integers $1, \ldots, n$ with parity ϵ_P. The functions $g_r(j, x)$ are as in (3.10*b*) and the wavenumbers are determined by solving the equations

$$\exp\left(\mathrm{i}q_r N + \mathrm{i}\sum_{t=1}^{n} \Theta(q_r, q_t)\right) = (-1)^{n-1}\omega^n \tag{3.11c}$$

for $r = 1, \ldots, n$. For this state we have

$$\mathscr{H}_{XXZ}\Psi(n; s) = (\epsilon_n - \tfrac{1}{4}N\Delta)\Psi(n; s), \tag{3.11d}$$

where

$$\epsilon_n = \sum_{r=1}^{n} (\Delta - \cos q_r). \tag{3.11e}$$

4. Discussion

Many formal aspects of the eigensolution sketched above are similar to the ordinary Bethe solution but in spite of superficial resemblances the two solutions are rather different. This is basically because the states $\psi_j(x_1, \ldots, x_n)$ are not eigenstates of S_z and the quantum number n is unrelated to S_z. Although the equations (3.11*c*) for the wavenumbers are very similar in form to those discussed by Yang and Yang (1966), the presence of the $L/2$th root of unity ω modifies the solutions in an interesting way. As an example, suppose the chain is of length N satisfying $N \equiv 2 (\mathrm{mod}\ L)$. Then we have $n = 1$ families and the corresponding eigenvectors are simple 'spin' waves in terms of the 'up' or 'down' spins of § 2.

In this case the wavenumbers q are determined by

$$\mathrm{e}^{\mathrm{i}qN} = \omega$$

or

$$\mathrm{e}^{\mathrm{i}qNL/2} = 1.$$

Thus these 'spin' waves have wavenumbers which one would not ordinarily associate with a simple spin chain of length N. The question of how to transform the solution sketched here into the familiar Bethe solution (and vice versa) is not one which can be easily answered. The wavefunctions in the two solutions are quite different. Thus a Baxter eigenvector such as (3.11*a*) with a fixed value of n is some superposition of Bethe eigenstates with all possible eigenvalues of S_z represented in the superposition. Since the spectrum of energies must be the same in both solutions it may be easier to look first for a simple relation between the wavenumbers of the Baxter solution as given by (3.11*c*) and those of the Bethe solution which are given formally by solving (3.11*c*) with $\omega = 1$ and the restriction (2.11*b*) on n removed. Any answer to this question must await further investigation.

A further interesting point is the occurrence of the parameter s in the eigenstates $\Psi(n; s)$ above. In Baxter's general eigenvectors for the eight-vertex model transfer matrix there are two parameters s and t. Our requirement of § 2 that the 'up' and 'down' spinors $\phi_{l,l+1}, \phi_{l,l-1}$ should be orthogonal has fixed t relative to s but otherwise our s is identical with Baxter's s. In (3.11*a*) we have included an explicit phase

$$\exp[-\mathrm{i}\tfrac{1}{2}s(N-2n)]$$

in front of the summation over $\psi_j(x_1, \ldots, x_n)$. If we look at

$$\exp[-\mathrm{i}\tfrac{1}{2}s(N-2n)]\psi_j(x_1, \ldots, x_n)$$

in detail then at site j we get a factor $\mathrm{e}^{-\mathrm{i}s/2}$ for an 'up' spin and a factor $\mathrm{e}^{\mathrm{i}s/2}$ for a 'down' spin,

$$\mathrm{e}^{-\mathrm{i}s/2}\phi_{l_j,l_j+1} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{\mathrm{i}2l_j\eta}\,\mathrm{e}^{\mathrm{i}s/2} \\ \mathrm{e}^{-\mathrm{i}s/2} \end{pmatrix} = \mathrm{e}^{\mathrm{i}\sigma_j^z s/2}\frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{\mathrm{i}2l_j\eta} \\ 1 \end{pmatrix},$$

$$\mathrm{e}^{+\mathrm{i}s/2}\phi_{l_j,l_j-1} = \mathrm{e}^{\mathrm{i}\sigma_j^z s/2}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -\mathrm{e}^{-\mathrm{i}2l_j\eta} \end{pmatrix}.$$

We observe then that

$$\Psi(n; s) = \exp(\mathrm{i}sS_z)\Psi(n; 0). \tag{4.1}$$

Thus the occurrence of this free parameter in the XXZ case is associated with the fact that both S_z and n are conserved quantities. In the ordinary Bethe solution S_z is diagonal while in the Baxter solution sketched above the operator corresponding to n is diagonal. The operator whose eigenvalue is n will be itself dependent upon the parameter s so that despite the appearance of equation (4.1) this operator will not commute with S_z. It is interesting to speculate that in the general XYZ case and in the eight-vertex model there may again be two conserved but non-commuting quantities, one of them being Baxter's n, the other being some generalization of S_z. For the special case of the asymmetric XY model one can obtain the operator corresponding to Baxter's n explicitly (Jones 1974). Hence this speculation may be pursued further most easily in the XY model.

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